3. NILPOTENCY AND QUASI-REGULARITY

§3.1. Nilpotency

We now turn our attention to non-commutative rings. The classic example is the ring of $n \times n$ matrices. This ring differs from the ring of integers, and other Euclidean Rings, in several respects. For a start it is non-commutative. Secondly the cancellation law doesn’t hold for matrices. Indeed there exist non-zero matrices whose square is zero.

An element r in a ring is **nilpotent** if $r^n = 0$ for some n. If x, y are commuting nilpotent elements then xy is nilpotent, and by the Binomial Theorem, x + y is nilpotent. But the product of two nilpotent (These are left as exercises.)

A ring R is a **nil ring** if every element is nilpotent. Clearly a nil ring can’t have an identity element 1.

A stronger condition than the property of being a nil ring is being nilpotent. If R is a ring we define R^n to be the set of all sums of products of n elements of R, and define R to be **nilpotent** if $R^n = 0$ for some n.

Example 1: The ring of 3 × 3 matrices of the form:

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

is nilpotent.

Every nilpotent ring is clearly a nil ring. However, the converse doesn't hold.

Example 2: Let B = $\{u_x \mid 0 < x < 1\}$ be a set of symbols indexed by the open interval (0, 1) in $\mathbb{R}$ and let R be the commutative algebra over $\mathbb{R}$ with basis B, where $u_x u_y =$

$$\begin{cases} u_{x+y} \text{ if } x + y < 1 \\ \quad 0 \text{ otherwise} \end{cases}.$$

Since $u_x^n = 0$ if $nx \geq 1$, every element of R is nilpotent and so A is nil.

For all n let $U_n = u_{1/2}u_{1/4}u_{1/8}\ldots u_{1/2^n} = u_{1/2+1/4+\ldots+1/2^n} \neq 0$. Hence A is not nilpotent.

Theorem 1: If R has a non-zero nil left ideal then it has a non-zero nil right ideal (and vice versa).

Proof: Let L be a non-zero nil left ideal and let $0 \neq x \in$ L. Thus Rx is a nil left ideal and so xR is a nil right ideal. If xR = 0 then $\mathbb{Z}x$ is a nilpotent right ideal. ✋☺

Theorem 2: Every nilpotent right (left) ideal is contained in a nilpotent 2-sided ideal.

Proof: Let I be a nilpotent right ideal of R.

By induction $(\mathrm{I} + \mathrm{RI})^n \leq \mathrm{I}^n + \mathrm{RI}^n$ for all $n \geq 1$ and so I + RI is a nilpotent 2-sided ideal. ✋☺

Theorem 3: If $x\mathrm{R}$ is nilpotent then x lies in a nilpotent right ideal.

Proof: (Note that if R doesn't have a 1 it may well be the case that $x \notin x\mathrm{R}$.)

$\mathrm{I} = \mathbb{Z}x + x\mathrm{R}$ is a right ideal containing x.

Suppose $(x\mathrm{R})^n = 0$.

Then every element of I^{2n} is a sum of products, each with n factors with one of the following forms: $xrxs$, x^2r, xrx, x^2 for $r, s \in R$.

Since these are all in $x\mathrm{R}$, $\mathrm{I}^{2n} \le (x\mathrm{R})^n = 0$. ✋☺

§3.2. Quasi-Regularity

If R is a ring we define $\boldsymbol{x \circ y}$ to be $x + y + xy$.

Theorem 4: R is a semigroup with identity under $\circ$.

Proof: Closure is obvious. The associative law holds because

$$(x \circ y) \circ z = (x + y + xy) \circ z = (x + y + xy) + z + (x + y + xy)z$$

$$= x + y + z + xy + xz + yz + xyz$$

$$= x \circ (y \circ z).$$ ✋☺

The identity of this semigroup is 0.

Theorem 5: If x has both a left inverse and a right inverse, they are equal.

Proof: Suppose $x \circ y = 0 = z \circ x$. Then $y = 0 \circ y = (z \circ x) \circ y = z \circ (x \circ y) = z \circ 0 = z$.

We call this inverse the **quasi-inverse** of x and denote it by $\mathbf{x}^{(-1)}$ to distinguish it from the usual inverse.

An element x in a ring R is **right quasi-regular** (**RQR**) if there exists $y \in \mathrm{R}$ such that $x \circ y = 0$. Note that 0 is the identity element of the semigroup and so y is the inverse of x. It is called the **right quasi-inverse** of x. The ring R is defined to be **right quasi-regular** if all its elements are. Left quasi-regularity is defined similarly. A ring is simply **quasi-regular** if it is both left and right quasi-regular.

Example: Remember example 15 in chapter 1? We said we'd file it away for future reference. Let R = $\left\{\frac{2m}{2n+1}\right\}$ with $m, n \in \mathbb{Z}$ as a subset of $\mathbb{Q}$. Then R is a quasi-regular ring. Since $\frac{2m}{2n+1} \circ \frac{-2m}{2(m+n)+1} = 0$.

Theorem 5: If R is a commutative ring with 1 then $1 + x$ has an inverse under $\circ$ if and only if x is quasi-regular.

Proof: Suppose $1 + x$ has an inverse in R. We may write it as $1 + y$.
Then $1 = (1 + x)(1 + y) = 1 + x + y + xy = 1 + x \circ y$ and so $x \circ y = 0$.
The converse is easily checked. ✋☺

Theorem 6: Every nilpotent element is quasi-regular.
Proof: Suppose that $x^n = 0$. Then $y = -x + x^2 - x^3 \ldots$ is a finite sum and $xy - x = y$ and so $x \circ y = 0$, and clearly $y \circ x = 0$. ✋☺

An **idempotent** element is a non-zero element e such that $e^2 = e$. Clearly, if e is an idempotent then all powers of e are equal to e (‘idempotent’ = ‘equal powers’,)

Theorem 7: If e is idempotent then $-e$ is neither RQR or LQR.
Proof: Suppose e is idempotent and $-e$ is RQR. Then $-e + x - ex = 0$ for some x.
Then multiplying on the left by e we get $0 = -e + ex - ex = -e$ and so $e = 0$, a contradiction. Similarly if e is LQR. ✋☺

Theorem 8: The class of quasi-regular rings is closed under S, Q and P.
Proof:

EXERCISES FOR CHAPTER 3

Exercise 1: Prove that if x and y are commuting nilpotent elements of a ring then xy is nilpotent.

Exercise 2: Prove that if x and y are commuting nilpotent elements of a ring then $x + y$ is also nilpotent.

Exercise 3: Find two nilpotent 2×2 matrices whose sum is not nilpotent.

Exercise 4: Find two nilpotent 2×2 matrices whose product is not nilpotent.

Exercise 5: Prove that if xy is nilpotent then so is yx.

Exercise 6: Find all the idempotent 2×2 matrices.

SOLUTIONS FOR CHAPTER 3

Exercise 1: Suppose $x^m = y^n = 0$ and xy = yx. Let M = MAX(m,n).
Then $(xy)^M = x^M y^M = 0$.

Exercise 2: Suppose $x^m = y^n = 0$ and $xy = yx$.
Then $(x + y)^{m+n-1} = x^{m+n-1} + \binom{m+n-1}{1} x^{m+n-2}\, y + \ldots + \binom{m+n-1}{n-1} x^m y^{n-1} + \binom{m+n-1}{n} x^{m-1} y^n + \ldots \binom{m+n-1}{1} x y^{m+n-2} + y^{m+n-1} = 0$.

Exercise 3: If A = $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then $A^2 = B^2 = 0$ but $(A+B)^2 = I$.

Exercise 4: If A = $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then $A^2 = B^2 = 0$ but AB = $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ which is not nilpotent.

Exercise 5: If $(xy)^n = 0$ then $(yx)^{n+1} = y(xy)^n x = 0$.

Exercise 6: Let E = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be idempotent.

Then $E^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = E$.

Hence $\begin{cases} a^2 + bc = 1 \\ b(a + d) = b \\ c(a + d) = c \\ d^2 + bc = d \end{cases}$

Case 1: *b* = 0: Then $a^2 = a$, $d^2 = d$. So $a = 0$ or 1 and $d = 0$ or 1.

So $a + d = 0$, 1 or 2.

Case 1A: *a* + *d* = 0: Then $a = d = 0$ and from one of the above equations, $c = 0$. In this case E is the zero matrix and so is not idempotent.

Case 1B: *a* + *d* = 1: Then $a = 0$ and $d = 1$ or $a = 1$ and $d = 0$.

Thus E = $\begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$ for some c.

Case 1C: $a + d = 2$: Then $a = d = 1$ and, from the above equations, $b = c = 0$. Thus E is the identity matrix.

Case 2: b ≠ 0: The 4 equations then reduce to just two.

$$\begin{cases} a^2 + \mathrm{bc} = 1 \\ \mathrm{a} + \mathrm{d} = 1 \end{cases}$$

Thus E = $\begin{pmatrix} a & b \\ \frac{a(1-a)}{b} & 1-a \end{pmatrix}$

The idempotent 2 × 2 matrices are:

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ \frac{a(1-a)}{b} & 1-a \end{pmatrix}$ for all a, c and $b \neq 0$.